Chopper model of pattern recognition

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A simple model is proposed that allows an efficient storage and retrieval of random patterns. Also correlated patterns can be handled. The data are stored in an Ising-spin system with ferromagnetic interactions between all the spins and the main idea is to "chop" the system along the boundaries where the patterns differ.

I. INTRODUCTION

The work of Hopfield[1] has led to a renewed interest in methods that allow pattern storage and retrieval in *content*-addressable (i.e., associative) memories. However, all the models that have been proposed until now are marred by the presence of unwanted, spurious states that are mixtures of the original patterns and drastically reduce the efficiency. Because of this the ensuing analysis requires a fair amount of sophistication.

In this paper we present a simple model whose main characteristics can be understood without detailed study, whose efficiency is rather high and can be calculated straightforwardly, and whose spurious states allow an immediate interpretation in terms of the model's energy landscape. The model itself is introduced in Sec. II, its memory function and efficiency are analyzed in Sec. III, and its main features are discussed in Sec. IV. We hope that this paper may serve as an elementary introduction to the fascinating subject of neural networks and pattern recognition.

The only prerequisite for reading this paper is an elementary understanding of the Curie-Weiss model of ferromagnetism. Given a configuration $\mathbf{S}=\{S(i);\ 1\le i\le n\}$ of n Ising spins $S(i)=\pm 1$, the Curie-Weiss Hamiltonian is given by

$$H(\mathbf{S})=-Jn\left[n^{-1}\sum_{i=1}^{n}S(i)\right]^2 . \tag{1}$$

The thermodynamics of (1) is easily obtained[2] in the limit $n\to\infty$. For the moment we only need to know the two ground states, $S(i)=+1$ for all i or $S(i)=-1$. If n is finite, there is an energy barrier of height proportional to n between the two ground states.

We introduce the following dynamics.[1] An arbitrary spin, say $S(i)$, is picked out of a spin configuration $\mathbf{S}$. Through flipping the spin from $S(i)$ to $-S(i)$ we transform the configuration $\mathbf{S}$ into a new state $\mathbf{S}'$. If

$$\Delta E=H(\mathbf{S}')-H(\mathbf{S})<0 , \tag{2}$$

we keep $\mathbf{S}'$. If $\Delta E\ge 0$, we keep $\mathbf{S}$. We continue the procedure by picking another spin. And so on. This is what is called a zero-temperature Monte Carlo procedure. A spin is flipped only if energy is gained.

It is easy to see that if $k>N/2$ spins in (1) are up and the remaining $N-k$ are down, then the system evolves to the state with all spins up. If on the other hand, $k<N/2$, then the result is all spins down. We will use this simple argument repeatedly.

II. THE MODEL

Suppose we are given p random patterns

$$\xi_\alpha=\{\xi_{i\alpha};\ 1\le i\le N\},\quad 1\le\alpha\le p , \tag{3}$$

where the $\xi_{i\alpha}$ are independent, identically distributed random variables which assume the values ± 1 with equal probability. The independence assumption simplifies the argument but is by no means necessary; cf. Sec. V. For the sake of convenience, N is a large power of 2. The patterns ξ_α may be embedded into the phase space of N Ising spins. The spin system is given a dynamics via the Hamiltonian

$$H_N(\mathbf{S})=-\tfrac{1}{2}\sum_{i,j}J_{ij}S(i)S(j) . \tag{4}$$

The idea[1] is to choose the coupling constants (bonds) J_{ij} in such a way that the ξ_α are *attracting* fixed points for the dynamics (2). Of course, the basins of attraction and, hence, the efficiency, should be as large as possible. Here we take

$$J_{ij}=\prod_{\alpha=1}^{p}\left[\frac{1+\xi_{i\alpha}\xi_{j\alpha}}{2}\right]\equiv\prod_{\alpha=1}^{p}J_{ij}^{(\alpha)} . \tag{5}$$

We remind the reader that the $\xi_{i\alpha}$ are ± 1 so that the J_{ij} vanish if $\xi_{i\alpha}\xi_{j\alpha}=-1$ for some α and equal one if $\xi_{i\alpha}\xi_{j\alpha}=1$ for all α. The patterns which have to be retrieved are stored in the bonds.

What is the effect of the ansatz (5)? To see this we imagine that all spins are connected ferromagnetically (stage zero) and then add the $J_{ij}^{(\alpha)}$ one after the other. See also Fig. 1. We start with $\alpha=1$.

The N spins may be divided into *two* groups: those which have $\xi_{i1}=+1$ and those where $\xi_{i1}=-1$. The interaction $(1+\xi_{i1}\xi_{j1})/2$ allows the spins in each group to interact ferromagnetically in the manner of Eq. (1) but

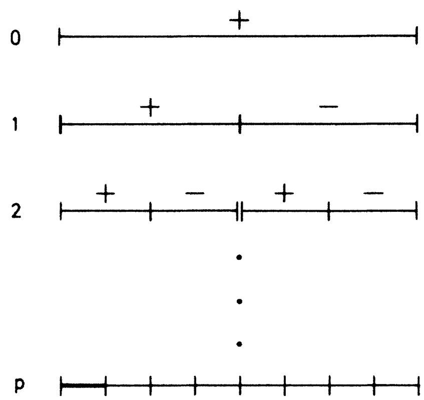

FIG. 1. Schematic representation of the partition of the index set $\{1 \le i \le N\}$ induced by the $\xi_{i\alpha}$. Start with the zeroth stage where all $\frac{1}{2}N(N-1)$ pairs are connected by ferromagnetic bonds of strength 1. Upon introducing the ξ_{i1}, we get a division of the index set into two groups: One where $\xi_{i1}=+1$ and another one where $\xi_{i1}=-1$. Imagine that the spins are reordered as indicated (+ to the left of −). The ξ_{i2} induce an analogous partition in each of these groups. This gives four of them. And so on. At the pth stage (here $p=3$) we end up with 2^p groups. Group to the most left has $\xi_{i\alpha}=+1$ for all i in the group and all α. Each group is uniquely characterized by a sequence of p plus and minus signs. The groups do not interact and the intragroup interaction is ferromagnetic as in the manner of Eq. (1).

there is no interaction *between* the two groups. About half of the ξ_{i1} are positive and about half of them are negative, so both groups have nearly the same size. We can reorder the spins so that the + spins are on the left and the − spins are on the right; cf. Fig. 1.

We now turn to the second stage and introduce the ξ_{i2}. Among the + spins there are approximately as many with $\xi_{i2}=+1$ as with $\xi_{i2}=-1$. Since $J_{ij}^{(2)}=(1+\xi_{i2}\xi_{j2})/2$, both groups interact ferromagnetically among themselves but there is no interaction between them. The same holds true for the − spins. As depicted in Fig. 1, we now have *four* noninteracting groups of spins. And so on, until we have built up the interaction (5), each time chopping the system along the boundaries where the patterns differ. Whence the name of the model. In general, a group is a set of indices i such that, whatever α, all the $\xi_{i\alpha}$ have the *same* sign. So it may be characterized by p plus and minus signs, depending on whether $\xi_{i\alpha}$ is +1 or −1, $1 \le \alpha \le p$.

There are 2^p groups of approximately

$$x = 2^{-p}N \tag{6}$$

spins each, the intragroup interaction is ferromagnetic whereas the intergroup interaction is absent. We have obtained, so to speak, an ultrametrically *de*coupled family. By construction, each ξ_α is a *ground state* of the Hamiltonian (4) with the interaction (5). Compare, for instance, the stage $\alpha=2$ in Fig. 1. This is the key to the memory function of the model. At whatever temperature, its statistical mechanics is trivial.

Given N, there is a theoretical upper bound for p. Combining Eq. (6) with the requirement $x \ge 1$ we get $p=\log_2 N$ for the maximal value of p. But this is somewhat academic as we will see shortly. We henceforth assume $x \gg 1$, unless stated otherwise. In taking the thermodynamic limit $N \to \infty$ we may either fix p or the mean group size $x=2^{-p}N$. If we take the second possibility and fix x, then the number p of stored patterns may increase with N (though rather slowly). It is implicitly understood that the J_{ij} are correctly normalized. One has to multiply (5) by N^{-1} if p is fixed and by x^{-1} if one fixes x.

III. MEMORY FUNCTION

External noise perturbs the patterns. This may be modeled by multiplying each $\xi_{i\alpha}$, $1 \le i \le N$, by a random phase η_i which assumes the values −1 and 1 with probabilities q and $1-q$, respectively. The η_i are assumed to be independent. The new state,

$$\xi'_\alpha = \{\xi_{i\alpha}\eta_i;\ 1 \le i \le N\}\ , \tag{7}$$

is taken as a starting point for the dynamics (2) and the question is under what condition the system relaxes to the original pattern ξ_α.

By (6), each group contains x spins on the average. Its dynamics is governed by a Hamiltonian of the form (1). Moreover, all groups are *independent.* If the thermodynamic limit $N \to \infty$ is taken with p fixed, then $x \to \infty$ and, as we have seen in Sec. I, each $q < \frac{1}{2}$ is allowed. With certainty the system then relaxes to the original pattern. In practical situations, however, N and x are finite. Then we require q to be such that the probability $\mathbf{P}$ of flipping more than half of the spins in each group is less than a given error bound Δ,

$$\mathbf{P} = \sum_{n \ge x/2}^{x} q^n (1-q)^{x-n} \begin{bmatrix} x \\ n \end{bmatrix} < \Delta\ . \tag{8}$$

This is the relevant probability estimate; cf. Fig. 2. A *whole* group either relaxes back to where it came from or

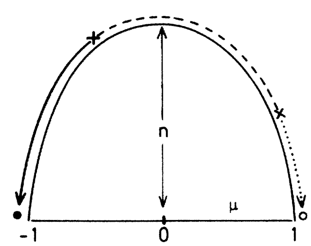

FIG. 2. The energy barrier associated with the Hamiltonian $H=-n[n^{-1}\sum_{i=1}^{n} S(i)]^2 \equiv -n\mu^2$ of Eq. (1). The energy (vertical scale) has been plotted as a function of the magnetization μ. The dynamics of each group, which is governed by such a Hamiltonian, only allows a *de*crease in energy. We suppose the starting point is on the left (solid circle). If the noise level q is not too high, the system will remain on the left and relax back to its starting point (solid line). However, either by too much noise or by thermal activation, the system may cross the hill (dashed line) and is then bound to relax to the right (dotted line). The information has been lost.

has "crossed the hill" and falls into the wrong valley. We may evaluate **P** by using the De Moivre-Laplace theorem,[3]

$$\sum_{n=x/2}^{x} q^n(1-q)^{x-n}\begin{bmatrix} x \\ n \end{bmatrix} \approx \int_a^b \frac{dy}{\sqrt{2\pi}} e^{-(1/2)y^2} \quad (9)$$

with

$$a=\frac{x/2-qx}{\sqrt{xq(1-q)}},\quad b=\frac{x-qx}{\sqrt{xq(1-q)}} . \quad (10)$$

The right-hand side of (9) has been tabulated.[4] In passing we note that, for a given probability **P**, there are $\mathbf{P}2^p x=\mathbf{P}N$ out of N spins that have the wrong sign (compared to the original pattern). These also make up a fraction **P** of the total number of spins.

What happens if n, the number of spins in a group, is rather small? The average value of n is x, which is given by (6). One spin cannot relax to its original state if it has been flipped, so we should take x larger than one. But if we take two spins and flip one of them, then with probability $\frac{1}{2}$ the dynamics (2) will drive the system to the wrong valley. We, therefore, take an average of at least four spins per group ($x=4$). The maximal value of p then follows from $x_{\min}=4=2^{-p}N$ which gives

$$p_{\max}=\log_2 N-2 . \quad (11)$$

One also has to realize that x is a *mean* value. For small x, the fluctuations of the group size n become important. In the Appendix we show that n is a random variable with a Poisson distribution, i.e.,

$$\mathrm{Prob}(n)=e^{-x}\frac{x^n}{n!} , \quad (12)$$

where Prob(n) is the probability there are n spins in a group. According to (12), the group size n has mean x, as should be the case, and standard deviation $\sqrt{x}$; cf. Ref. 5. Now take $x=4$. Then $\mathrm{Prob}\{n\leq 2\}=0.238$. For $x=9$, however, this probability is already very small (0.006). The probabilities associated with the Poisson distribution have also been tabulated.[4]

IV. DISCUSSION

Given p, there are 2^p groups of about x spins each. The spins in a group, which are coupled ferromagnetically, can be in only two ground states. For 2^p groups this makes 2^{2^p} ground-state configurations and, hence, an equal number of energy valleys which are separated by (free-) energy barriers of height proportional to x as soon as the temperature T is somewhat below the critical temperature T_c of the Curie-Weiss ferromagnet ($x\gg 1$). Only p configurations out of this huge number of ground states are relevant: the p original patterns ξ_α we started with.

The remaining ground states are spurious. In spite of their huge number, noisy patterns are recognized with extremely small error if x is large enough. It is to be noted, though, that heating up followed by cooling down does not remove the remaining faults. The faulty groups may make a thermal jump over the energy barrier so as to land in the original valley but other groups may migrate with equal probability to the opposite valley (cf. Fig. 2) and, hence, introduce new errors.

Finally, we would like to point out the relation between the present work and a recent prescription proposed by Kinzel.[6] Kinzel starts with the Sherrington-Kirkpatrick model where the J_{ij} are all independent Gaussians with mean zero and variance N^{-1}. Then the J_{ij} with $\xi_{i\alpha}J_{ij}\xi_{j\alpha}<0$ for some α, that is, the frustrated J_{ij}, are deleted. The procedure is a special case of what we have done here. To see this, we perform the gauge (Mattis) transformation $S(i)\to\xi_{i1}S(i)$. Then the bonds J_{ij} are changed into $J'_{ij}=\xi_{i1}J_{ij}\xi_{j1}$, the patterns are given by $\xi'_{i\alpha}=\xi_{i\alpha}\xi_{i1}$, and a bond is removed if $J'_{ij}<0$ ($\alpha=1$) or $\xi'_{i\alpha}\xi'_{j\alpha}=-1$ ($\alpha\geq 2$). After the first stage $\alpha=1$ we are left with ferromagnetic bonds, which are cut if $\xi'_{i\alpha}\xi'_{j\alpha}=-1$, i.e., if they connect antiparallel spins for some pattern $\alpha\geq 2$—as in Sec. II.

V. SUMMARY

The present model, though extremely simple, has all the properties of a content-addressable memory. If the group size is large enough, patterns are readily recognized. Moreover, the network can easily learn patterns and it is *not* necessary that they are random (uncorrelated). The independence serves to guarantee that the different groups have about equal size. The only proviso to limit errors is that the different groups have a reasonable size. It is a matter of elementary statistics to estimate this for a given noise level q.

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APPENDIX

We want to determine the probability distribution of the group size n as $N\to\infty$ and $2^{-p}N=x$ is fixed. It is to be shown that n has a Poisson distribution,

$$\mathrm{Prob}(n)=e^{-x}\frac{x^n}{n!} , \quad (\mathrm{A1})$$

with characteristic function

$$c(t)=\mathscr{E}\{e^{itn}\}=\sum_{n=0}^{\infty} e^{-x}\frac{x^n}{n!}e^{itn}=\exp[x(e^{it}-1)] \quad (\mathrm{A2})$$

$\mathscr{E}\{\ \}$ denotes a mathematical expectation. A characteristic function determines the probability distribution completely.

Since the $\xi_{i\alpha}=\pm 1$ are independent, identically distributed random variables, it suffices to pick a specific group, say the one most to the left at level p in Fig. 1 where $\xi_{i\alpha}=+1$ for all i in the group and all α. Our task is to show that the group size has the distribution (A1).

Let $X_{i\alpha}=(1+\xi_{i\alpha})/2$, $1\leq i\leq N$, be the random variable that equals one if $\xi_{i\alpha}=1$ and vanishes if $\xi_{i\alpha}=-1$. Furthermore, let

$$Y_i = \prod_{\alpha=1}^{p} X_{i\alpha} \tag{A3}$$

and

$$S_N = \sum_{i=1}^{N} Y_i \ . \tag{A4}$$

S_N gives the size of the specific group under consideration.

We now show

$$\lim_{N\to\infty} \text{Prob}(S_N = k) = e^{-x}\frac{x^k}{k!} \ . \tag{A5}$$

The proof is simple. We calculate the characteristic function $c_N(t)$ and show that $c_N(t) \to c(t)$ of (A2) as $N \to \infty$. To this end we first note that either $Y_i = 1$ or 0 and

$$\text{Prob}(Y_i = 1) = \prod_{\alpha=1}^{p} \text{Prob}(X_{i\alpha} = 1) = 2^{-p} \ , \tag{A6}$$

while $\text{Prob}(Y_i = 0) = 1 - 2^{-p}$. By independence and this simple observation,

$$\begin{aligned} c_N(t) &= \mathscr{E}\{\exp(itS_N)\} = \prod_{j=1}^{N} \mathscr{E}\{\exp(itY_j)\} \\ &= [1 + 2^{-p}(e^{it} - 1)]^N \ , \end{aligned} \tag{A7}$$

and since $2^{-p} = x/N$ we find, as $N \to \infty$,

$$c_N(t) = [1 + (x/N)(e^{it} - 1)]^N \to \exp[x(e^{it} - 1)] \ , \tag{A8}$$

as advertised.

[1]J. J. Hopfield, Proc. Natl. Acad. Sci. USA **79**, 2554 (1982).

[2]M. Kac, in *Statistical Physics, Phase Transitions and Superfluidity,* edited by M. Chrétien, E. P. Gross, and S. Deser (Gordon and Breach, New York, 1968), Vol. 1, pp. 248 and 249. See also, H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Oxford University Press, Oxford, 1971), Sec. 6.5.

[3]K. L. Chung, *Elementary Probability Theory and Stochastic Processes* (Springer-Verlag, New York, 1974), Sec. 7.3, Theorem 5.

[4]See, Ref. 3, Table 1, or B. V. Gnedenko, *The Theory of Probability* (Chelsea, New York, 1968). Gnedenko's book also contains tables for the Poisson distribution.

[5]Reference 3, Sec. 7.1.

[6]W. Kinzel, Z. Phys. B **60**, 205 (1985).