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Isometric and unitary phase operators: explaining the Villain transform

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Abstract
The Villain transform plays a key role in spin-wave theory, a bosonization of elementary excitations in a system of extensively many Heisenberg spins. Intuitively, it is a representation of the spin operators in terms of an angle and its canonically conjugate angular momentum operator and, as such, has a few nasty boundary-condition twists. We construct an isometric phase representation of spin operators that conveys a precise mathematical meaning to the Villain transform and is related to both classical mechanics and the Pegg–Barnett–Bialynicki-Birula boson (photon) phase operators by means of suitable limits. In contrast to the photon case, unitary extensions are inadequate because they describe the wrong physics. We also discuss in some detail the application to spin-wave theory, pointing out some examples in which the isometric representation is indispensable.

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1. Introduction

The Villain transform [1] is a representation of the quantum spin operators in terms of an angle (phase) and its canonically conjugate angular momentum operator. It is a mainstay to spin-wave theory and related to a phase representation of creation and annihilation operators of bosons (photons) introduced by Bialynicki-Birula [2], as explained below.

In classical physics a localized or convergent light beam is obtained by superposing plane waves with well-defined phase relations. Since Dirac's 1927 paper [3], this basic role of the phase has not ceased to motivate the search for a unitary phase operator '$e^{i\varphi}$' in quantum mechanics [2–5]. Recent rigorous work [6] on the interaction of N-level atoms with a quantized electromagnetic field has clarified the relationship between the phase operator and the classical limit of the field under which a suitably defined average number of photons tends to infinity while the photon density remains fixed.

In the present paper we introduce a phase operator formalism for quantum spins that, through suitable limits, is related to both the work of Guérin *et al* [6] and classical physics. As an application, we provide a precise mathematical definition of the Villain transform, whose original intuitive definition was flawed by boundary-value problems. A different insight into the problem of unitary extensions of the phase operators follows naturally from our treatment.

We also discuss in some detail the application to spin-wave theory, pointing out some examples in which the isometric representation is indispensable.

2. Mathematical formalism

In the approach of Guérin *et al* [6] a phase representation of the atom-field Hamiltonian due to Bialynicki-Birula [2] is used. The authors are thereby able to construct an isomorphism between the one-particle Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathrm{d}x)$, generated by the harmonic-oscillator basis $\{|n\rangle; n = 0, 1, 2, \ldots\}$, and the space $\mathcal{L}_{\overline{n},\theta}$ defined as a subspace of $\mathcal{L} \equiv (\mathcal{S}^1, \mathrm{d}\theta/2\pi)$, the square-integrable periodic functions of the angle θ, i.e., on the circle S^1, generated by the basis functions $\{|k\rangle = |\mathrm{e}^{\mathrm{i}k\theta}\rangle; -\overline{n} \leqslant k < \infty\}$. That is,

$$L_2(\mathcal{S}^1, \mathrm{d}\theta/2\pi) = \mathcal{L} \supseteq \mathcal{L}_{\overline{n},\theta} = [\{|k\rangle = |\mathrm{e}^{\mathrm{i}k\theta}\rangle; -\overline{n} \leqslant k < \infty\}]^c, \tag{1a}$$

where $[\cdots]^c$ denotes the closed linear span of $(\cdots)$.

In the limit $\overline{n} \to \infty$ one obtains the whole space $L_2(\mathcal{S}^1, \mathrm{d}\theta/2\pi)$. Identifying $\mathcal{H}$ as the space of the one-photon mode of the electromagnetic field, we see this limit [6] to correspond to a 'mean-photon number' going to infinity. By this isomorphism, the creation, annihilation and photon number operators (a^+, a and $N = a^+a$) have the following representation as operators acting on $\mathcal{L}_{\overline{n},\theta}$:

$$a^+_{\overline{n},\theta} = \sqrt{\overline{n} - \mathrm{i}\frac{\partial}{\partial\theta}}\,\mathrm{e}^{\mathrm{i}\theta}\,P_{\overline{n}} \tag{1b}$$

$$a_{\overline{n},\theta} = \mathrm{e}^{-\mathrm{i}\theta}\sqrt{\overline{n} - \mathrm{i}\frac{\partial}{\partial\theta}}\,P_{\overline{n}} \tag{1c}$$

$$N_{\overline{n},\theta} = \left(\overline{n} - \mathrm{i}\frac{\partial}{\partial\theta}\right) P_{\overline{n}}, \tag{1d}$$

where

$$P_{\overline{n}} = \sum_{k=-\overline{n}}^{\infty} |\mathrm{e}^{\mathrm{i}k\theta}\rangle\langle\mathrm{e}^{\mathrm{i}k\theta}| \tag{1e}$$

is the projector of $L_2(\mathcal{S}^1, \mathrm{d}\theta/2\pi)$ onto $\mathcal{L}_{\overline{n},\theta}$.

When equations (1*b*)–(1*d*) are written for each mode $(\boldsymbol{k}, \alpha)$ of the electromagnetic field, where α denotes the polarization and $\boldsymbol{k}$ the wavevector, then the Fourier expansion of the (quantum) vector potential operator in terms of plane waves ([15], (2.87), p 34) leads us to interpret each $\theta_{\boldsymbol{k},\alpha}$ in (1*b*), (1*c*) as the observable corresponding to the phase of the plane wave in question in the sense of classical optics. The coefficients $\sqrt{N_{\boldsymbol{k},\alpha}} = [\overline{n} - \mathrm{i}\partial/\partial\theta_{\boldsymbol{k},\alpha}]^{1/2}$ in (1*b*) and (1*c*)s correspond to the amplitude of the wave in classical optics, because $(\sqrt{N_{\boldsymbol{k},\alpha}})^2 = N_{\boldsymbol{k},\alpha}$, which, by (1*d*), represents the number of photons in the mode $(\boldsymbol{k}, \alpha)$, i.e., the intensity. Finally, since $-\mathrm{i}\partial/\partial\theta_{\boldsymbol{k},\alpha}$ assumes positive and negative values, $N_{\boldsymbol{k},\alpha} = \overline{n} - \mathrm{i}\partial/\partial\theta_{\boldsymbol{k},\alpha}$, i.e., the photon number, is computed 'around' a large 'mean photon number $\overline{n}$' [6], in agreement with the semiclassical limit ([15], pp 35–37).

Because of (1) we clearly see that the phase operator '$\mathrm{e}^{\mathrm{i}\theta}$' is not unitary since its range is $\mathcal{L}_{\overline{n}+1,\theta} \neq \mathcal{L}_{\overline{n},\theta}$. Thus the rigorous 'phase representation' (1*b*) and (1*c*) of the operators a and a^+ involves necessarily *non*-unitary phase operators $\mathrm{e}^{-\mathrm{i}\theta}$ and $\mathrm{e}^{\mathrm{i}\theta}$. They are, however, even by definition *isometric* since they *preserve* the norm in $\mathcal{L}_{\bar{n},\theta}$: $\|f\|^2 = \|e^{\pm i\theta} f\|^2$ where $\|f\|^2 \equiv \sum_{k=-\bar{n}}^{\infty} |f_k|^2$ with $f(\theta) \equiv \sum_{k=-\bar{n}}^{\infty} f_k\, e^{ik\theta}$.

Pegg and Barnett [5] pointed out that unitary phase operators may be defined on finite-dimensional subspaces of $\mathcal{H}$. The action of the corresponding approximate annihilation and creation operators mimics that of the conventional annihilation and creation operators if the state of interest is 'physically accessible', i.e., orthogonal to highly excited number states [5, 7]. Meanwhile the connection between the Pegg–Barnett [5] and the Bialynicki-Birula phase operator [2] is known [7].

3. Quantum-mechanical spins and Villain transform

In this note we revisit the representation analogous to (1) for a quantum spin $\boldsymbol{S} := (S_x, S_y, S_z)$ of finite spin quantum number S, satisfying the $SU(2)$ commutation relations (we set $\hbar = 1$):

$$[S_x, S_y] = iS_z, \qquad \text{et cyclic.} \tag{2}$$

Identifying $\bar{n}$ in the above-mentioned formalism for photons with S, we obtain the basis phase states in the $(2S + 1)$-dimensional subspaces of the unified Pegg–Barnett–Bialynicki-Birula approach {(7), equations (15) and (16)}. It turns out, however, that in the quantum spin context these phases have a natural physical interpretation in terms of the so-called Villain transform [8]. In contrast to the photon case, the unitary phases are inadequate to describe the physics of the system, as we shall see.

A quantum spin (2) does not behave like a particle, except under very special conditions that underline the so-called *diffusion approximation*; see for example [8]. In several applications, e.g., the classical theory of spin waves [9], it is important to devise a Hamiltonian formalism for one and, hence, $N < \infty$ *classical* spins, so that we assume we are given a function $\mathcal{H}$ (the Hamiltonian) of the classical spin components denoted here by a tilde $\widetilde{S}_x$, $\widetilde{S}_y$ and $\widetilde{S}_z$, and wish to find a canonical pair (q, p) in such a way that Hamilton's equations $\dot{q} = \partial\mathcal{H}/\partial p$ and $\dot{p} = -\partial\mathcal{H}/\partial q$ hold. Considering $\widetilde{S}_x$, $\widetilde{S}_y$ and $\widetilde{S}_z$ as functions of the (yet to be introduced q and p), the necessary and sufficient condition to be satisfied by (q, p) is that

$$\{q, p\} = 1 \tag{3}$$

and that the Poisson bracket, in correspondence to (2), equals

$$\begin{aligned}\{\widetilde{S}_x, \widetilde{S}_y\} &\equiv \frac{\partial \widetilde{S}_x}{\partial q}\frac{\partial \widetilde{S}_y}{\partial p} - \frac{\partial \widetilde{S}_x}{\partial p}\frac{\partial \widetilde{S}_y}{\partial q} \\ &= \widetilde{S}_z, \qquad \text{et cyclic.}\end{aligned} \tag{4}$$

Setting

$$q = \widetilde{S}_z \tag{5}$$

so that q is the z-component of the classical spin and

$$p = -\phi \tag{6}$$

and where ϕ is the azimuth, it follows from (5) and (6) that

$$\widetilde{S}_x = (\sigma^2 - q^2)^{1/2} \cos p \tag{7}$$

$$\widetilde{S}_y = (\sigma^2 - q^2)^{1/2} \sin p, \tag{8}$$

where

$$\sigma^2 \equiv \widetilde{S}_x^2 + \widetilde{S}_y^2 + \widetilde{S}_z^2 \tag{9}$$

is the classical value of the total spin. Then (3) is satisfied by construction and (4) may be easily verified. It thus looks as if p and q describe a classical particle and indeed they do.

One may naturally ask what is the quantum-mechanical analogue of the above classical picture. It was introduced by Villain [1] but, in order to give it a precise mathematical meaning, it is necessary to discuss the boundary conditions in detail. Performing the classical canonical transformation $(q, p) \mapsto (-p, q)$ in (5) and (6), we arrive at

$$\varphi = \hat{q} \qquad \hat{S}_z = -\mathrm{i}\frac{\partial}{\partial\varphi} \equiv -\mathrm{i}\partial_\varphi \tag{10}$$

as canonically conjugate operators (in a formal sense) acting on the finite-dimensional Hilbert space

$$\widetilde{\mathcal{L}}_{S,\varphi} = \left[\varphi_m^S(\varphi); -S \leqslant m \leqslant S\right], \tag{11}$$

where $[\psi_m]$ denotes linear span of the functions ψ_m and the scalar product for $\psi^S, \phi^S \in \widetilde{\mathcal{L}}_{S,\varphi}$ is

$$(\psi^S, \phi^S) = \int_0^{2\pi} \mathrm{d}\varphi \overline{\psi}^S(\varphi)\phi^S(\varphi). \tag{12a}$$

Here we have set

$$\varphi_m^S \equiv (2\pi)^{-1/2} \exp(\mathrm{i}m\varphi) \tag{12b}$$

with m integer or half integer, depending on S being integer or half-integer, as usual.

Let

$$S_\pm = S_x \pm \mathrm{i}S_y. \tag{13}$$

Our version of the Villain representation is

$$S_+ = U\sqrt{S(S+1) - \hat{S}_z(\hat{S}_z + 1)} \tag{14a}$$

$$S_- = \widetilde{U}\sqrt{S(S+1) - \hat{S}_z(\hat{S}_z - 1)} \tag{14b}$$

where, on $\widetilde{\mathcal{L}}_{S,\varphi}$,

$$U = \begin{cases} \mathrm{e}^{\mathrm{i}\varphi} & \text{on} \quad (\varphi_S^S)^\perp \\ 0 & \text{on} \quad \varphi_S^S \end{cases} \tag{15}$$

and

$$\widetilde{U} = \begin{cases} \mathrm{e}^{-\mathrm{i}\varphi} & \text{on} \quad (\varphi_{-S}^S)^\perp \\ 0 & \text{on} \quad \varphi_{-S}^S, \end{cases} \tag{16}$$

where $\exp(\pm\mathrm{i}\varphi)$ are multiplicative operators with $0 \leqslant \varphi < 2\pi$; the operator $\hat{S}_z$ is given by (10).

Polar decomposition. The key idea underlying our proof is that of a 'polar decomposition' of an operator on Hilbert space. We refer to Reed and Simon [10] for the mathematical details and quickly sketch the underlying arguments. The polar decomposition is the generalization of $z = \exp(\mathrm{i} \arg z)|z|$ of a complex number z to an operator A where even its Hermitian conjugate or adjoint A^* cannot be expected to commute with A. The operator A^*A is a positive one so that we can take its (unique) positive square root $|A| := \sqrt{A^*A}$. According to the definition of $|A|$ we get

$$\||A|\psi\|^2 = (\psi, |A|^2\psi) = (\psi, A^*A\psi) = (A\psi, A\psi) = \|A\psi\|^2.$$

Let ker A denote the linear span of all vectors ψ with $A\psi = 0$ and let ran A, the linear span of all vectors $A\phi$, be the range of A; if necessary, one can add their closure.

The above equation tells us two things. First, $\ker A = \ker|A|$. In general, $\ker A \neq 0$ and $\mathcal{H} = \ker|A| \oplus \operatorname{ran}|A|$. Second, aiming at a polar decomposition of the form $A = U|A|$ we therefore cannot expect U to be unitary. Instead it is a *partial* isometry 'unitarily', i.e., 1-1, mapping $\operatorname{ran}|A|$, the orthogonal complement of $\ker|A|$, onto ran A. That is, we define U by $U(|A|\psi) := A\psi$ and directly see it is a well-defined partial isometry from $\operatorname{ran}|A|$ onto ran A. Finally, we extend U to all of $\mathcal{H}$ by putting it zero on $\{\operatorname{ran}|A|\}^\perp = \ker|A|$, the orthogonal complement of $\operatorname{ran}|A|$, and we are done.

In passing we note that in our case the Hilbert space $\mathcal{H}$ is the finite-dimensional space $\mathbb{C}^{2S+1} = [|m\rangle, -S \leqslant m \leqslant S]$, isomorphic to $\widetilde{\mathcal{L}}_{s,\varphi}$ as given by (11). Incidentally, there is an explicit general formula for U ([10], chapter VII, problem 20): $U = s - \lim_{n\to\infty} Af_n(|A|)$ where f_n is defined by $f_n(x) = 1/x$ if $x \geqslant 1/n$ and $f_n(x) = 1/n$ if $x \leqslant 1/n$.

The proof of (14)–(16) now follows. Let us write

$$S_+ = U|S_+|. \tag{17}$$

S_+ is trivially closed because it is defined on a finite-dimensional space. By the theory of angular momentum,

$$|S_+| = (S_-S_+)^{1/2} = [S(S+1) - S_z(S_z+1)]^{1/2}. \tag{18}$$

Now $\mathcal{H}_- \equiv \operatorname{ran}|S_+| = (\ker|S_+|)^\perp = [|-S\rangle, \ldots, |S-1\rangle]$ in terms of the orthonormal basis $\{|m\rangle; m = -S, \ldots, S\}$ of eigenvectors of S_z in $\mathbb{C}^{2S+1}$

$$S_z|m\rangle = m|m\rangle \tag{19}$$

Alternatively it may be expressed in terms of the operator S_z in $\widetilde{\mathcal{L}}_{s,\varphi}$. We note U does map $\mathcal{H}_-$ into ran $S_+ = (\ker S_-)^\perp = [|-S+1\rangle, \ldots, |S\rangle] \equiv \mathcal{H}_+$ because $S_- = S_+^*$ by (13). From (18) we also see that $|S_+|$ and S_z commute. Hence, from (18) and the relation

$$[S_z, S_+] = S_+ \tag{20}$$

it follows that

$$[S_z, U]|S_+| = U|S_+|. \tag{21}$$

Now, again by (18), $|S_+|$ annihilates the vector $|S\rangle$, whence, from (21) and the uniqueness of the polar decomposition (17), we obtain

$$[S_z, U] = U \quad \text{on} \quad \{|S\rangle\}^\perp. \tag{22}$$

We thus define the basic commutation algebra of our system, which would be formally guessed from (10), by

$$[S_z, U] = U. \tag{23}$$

By (19) and (22), and the fact that U is uniquely determined we obtain

$$U|m\rangle = |m+1\rangle, \qquad -S \leqslant m \leqslant S-1 \tag{24}$$

which proves (15) on $\widetilde{\mathcal{L}}_{S,\varphi}$. By (18) and (19) and the isomorphism between $\mathbb{C}^{2S+1}$ and $\widetilde{\mathcal{L}}_{S,\varphi}$ we arrive at (14*a*). Equations (14*b*) and (16) may be obtained in a similar way.

We now come to (15) and (16). Since U *follows* $|S_+|$ in definition (18) of S_+ it does not matter for the definition of S_+ how $U|S\rangle$ is defined. If we set

$$U_1|S\rangle = \mathrm{e}^{+\mathrm{i}\alpha}|-S\rangle \qquad \text{with} \qquad 0 \leqslant \alpha < 2\pi \tag{25a}$$

we arrive at a one-parameter family of unitary *extensions* U_1 of U. Correspondingly, we must set

$$\widetilde{U}_1|-S\rangle = \mathrm{e}^{-\mathrm{i}\alpha}|-S\rangle \qquad \text{with} \qquad 0 \leqslant \alpha < 2\pi. \tag{25b}$$

Adopting the representation (14) we need the basic commutation algebra (23) to show in particular that S_- as given by (14*b*) and (16) equals S_+^* with S_+ given by (14*a*). It is immediate, however, that the unitary extensions (25) do *not* preserve the basic commutation algebra,

$$\begin{aligned}[S_z, U_1]|S\rangle &= \mathrm{e}^{\mathrm{i}\alpha}(-S - S)|-S\rangle \\ &= -2S\,\mathrm{e}^{\mathrm{i}\alpha}|-S\rangle \neq \mathrm{e}^{\mathrm{i}\alpha}|-S\rangle.\end{aligned} \tag{26}$$

Accordingly the only possible extension of U to the whole space that still preserves (23) is $U|S\rangle = 0$; similarly, $\widetilde{U}|-S\rangle = 0$. We have thus proven (14)–(16). In the next section we discuss in some detail an application to spin-wave theory, pointing out some examples in which the isometric representation cannot be dispensed with.

4. An example from spin-wave theory

We consider a (for simplicity) linear chain of spins $\vec{S}_l$, of spin quantum number S, with periodic boundary conditions after N spins, i.e., $\vec{S}_{l+N} = \vec{S}_l$. We assume that its dynamics is described by the Heisenberg Hamiltonian

$$H_N = -J\sum_{l=1}^{N} \vec{S}_l \cdot \vec{S}_{l+1} = \sum_{l=1}^{N} H_l \tag{27}$$

with $J > 0$, corresponding to ferromagnetism. The ground states (g.s.) of H_N are [13]

$$\Omega_N(\pm) = \bigotimes_{l=1}^{N} |\pm S\rangle_l. \tag{28}$$

In (27) we may write

$$H_l = -\vec{\omega}_l \cdot \vec{S}_l = \tfrac{-1}{2}(\omega_{+,l}S_{-,l} + \omega_{-,l}S_{+,l} + 2\omega_{z,l}S_{z,l}) \tag{29a}$$

with

$$\omega_{\pm,l} = \omega_{x,l} \pm \mathrm{i}\omega_{y,l} \tag{29b}$$

and

$$\vec{\omega}_l \equiv J(\vec{S}_{l-1} + \vec{S}_{l+1}). \tag{30}$$

The Heisenberg equations of motion resulting from (29) are

$$\frac{\mathrm{d}S_{+,l}}{\mathrm{d}t} = \mathrm{i}\omega_{+,l}S_{z,l} - \mathrm{i}\omega_{z,l}S_{+,l} \tag{31a}$$

$$\frac{\mathrm{d}S_{-,l}}{\mathrm{d}t} = -\mathrm{i}\omega_{-,l}S_{z,l} + \mathrm{i}\omega_{z,l}S_{-,l} \tag{31b}$$

$$\frac{\mathrm{d}S_{z,l}}{\mathrm{d}t} = -\frac{\mathrm{i}}{2}(\omega_{+,l}S_{-,l} - \omega_{-,l}S_{+,l}). \tag{31c}$$

We consider now very low temperatures and assume that we are close to **one** of the g.s. (28) (an external magnetic field will realize that choice): we choose $\Omega_N(-)$. Thus,

$$S_{z,l} \approx -S \quad \text{for all} \quad l. \tag{32}$$

The components $S_{l,+}$ and $S_{l,-}$ are therefore of first order of smallness. From (31) and (32) we obtain in first approximation:

$$\frac{dS_{+,l}}{dt} = -iJS(S_{+,l-1} + S_{+,l+1}) + 2iJSS_{+,l} \tag{33a}$$

$$\frac{dS_{-,l}}{dt} = iJS(S_{-,l-1} + S_{-,l+1}) - 2iJSS_{-,l} \tag{33b}$$

$$\frac{dSz,l}{dt} = 0. \tag{33c}$$

Writing, now, by (32), (14a) and (15),

$$S_{+,l} = U_l\sqrt{S(S+1) - S_{z,l}(S_{z,l}+1)} \approx \sqrt{2S}U_l \tag{34a}$$

Similarly, close to g.s. $\Omega_N(+)$,

$$S_{-,l} = \tilde{U}_l\sqrt{2S} \tag{34b}$$

requires (16b), otherwise $S_-|-S\rangle = 0$ is violated. As long as the states $|+S\rangle$ (in (34a)) and $|-S\rangle$ (in (34b)) are accessible, albeit with small probability for low temperatures, neglecting (15b) (respectively (16b)) will lead to errors, because the operators U_l (respectively$\tilde{U}_l$) are not followed by the square roots as in (14a) (respectively (14b)) which are automatically zero on $|+S\rangle$ (respectively $|-S\rangle$) . Thus, whenever approximations are performed on the square roots in (14a) and (14b), which destroy their property of being zero at the boundary vectors, the conditions of isometricity (15b) and (16b) are *necessary* in order to allow for the *kinematic interaction* , i.e., the fact that the spectrum of $S_{z,l}$ is restricted to the interval $[-S, S]$. Such approximations occur often when 'developing' the mentioned square roots in power series, assuming that only a few vectors around one of the ground states contribute in the calculation of thermal expectation values. For models with strong planar anisotropy, when the ground state is in the $S_{z,l} = 0$ subspace for all values of l, at very low temperatures, relevant configurations do not involve large values of $|S_{x,i}|$, and practical effects of the kinematic interaction are negligible for large S [16]. In this case, instead of (34) the relevant approximations would be (with the g.s. consisting of all spins aligned in the x (or y) direction instead of (28)):

$$S_{+,l} = U_l\sqrt{S(S+1)} \tag{35a}$$

$$S_{-,l} = \tilde{U}_l\sqrt{S(S+1)}. \tag{35b}$$

In these cases the linear spin-wave approximation seems to be very good even for small values of S: the exact energy of a spin one-half XY chain is only 5% off the spin-wave value [13], and Anderson [17] has shown that the exact ground state energy of a Heisenberg spin one-half antiferromagnet in the linear spin-wave approximation is only 3% off the exact value! In the latter case the ground state for the spin-wave approximation is chosen to be the (classical) Ising antiferromagnet. For such low values of S the states $|+S\rangle$ and $|-S\rangle$ have non-negligible overlap even at low temperatures, and our remarks on the necessity of considering the kinematic interaction grow considerably in relevance.

As an application of (34a) (the same result is obtained for (35a)), choose in (15a) the angle $\phi = \phi_l$ as

$$\phi_l = \omega t - kl. \tag{36}$$

Putting (36) and (34a) into (33a), we obtain

$$i\omega JS\sqrt{2S}\exp[i(\omega t - kl)] = -iJS(\exp-(ik) + \exp(ik) - 2)\sqrt{2S}\exp[i(\omega t - kl)] \tag{37}$$

from which we get

$$\omega = \omega_k = 2J(1 - \cos k) \tag{38}$$

i.e., the spin-wave dispersion relation. For antiferromagnets, this procedure is modified, because the expansion is around the g.s. of the Ising antiferromagnet, and a linear dispersion relation for small k results. The physical interpretation of the angle variable in the present application is clear: it is the phase of the spin waves.

5. Discussion

The above considerations provide a new insight into the issue of the existence/non-existence of a self-adjoint 'phase operator' in quantum mechanics. In contrast to the photon case, we cannot take, as in [5], S *arbitrarily* large and restrict ourselves to 'physically accessible' states, i.e., orthogonal to $|S\rangle$ since in the spin case S is a fixed number. That is, the fundamental reason why unitary extensions are, in general, inadequate is that they describe the wrong physics, i.e., they do not preserve the basic commutation algebra, which in our case is (23)!

We conclude with some remarks relating our construction to both the starting point, i.e., classical physics as described by (5)–(8), and the boson representation (1). In the classical limit, the operators (13) and (19) tend [11] to c-number functions on the sphere

$$\frac{S_+}{S} \underset{S\to\infty}{\longrightarrow} \widetilde{S}_+ = \mathrm{e}^{\mathrm{i}q}\sqrt{1-p^2} \tag{39a}$$

$$\frac{S_-}{S} \underset{S\to\infty}{\longrightarrow} \widetilde{S}_- = \mathrm{e}^{-\mathrm{i}q}\sqrt{1-p^2} \tag{39b}$$

$$\frac{S_z}{S} \underset{S\to\infty}{\longrightarrow} \widetilde{S}_z = q. \tag{39c}$$

The proper sense in which the above limits (39) are to be understood has been described elsewhere [11, 12].

Division by S in (39) implies the vanishing of the commutators in the formal limit $S \to \infty$, as required in the classical limit. By a different contraction, viz., the so-called Lévy–Nahas contraction [12], $SU(2)$ goes over into the Lie algebra of the Heisenberg or oscillator group (1): the operators $(2S)^{-1/2}S_+$, $(2S)^{-1/2}S_-$ and $S^{-1/2}S_z$ tend (in a suitable sense [12]) to a^+, a and (-1), respectively, where a^+ (respectively a) are boson creation (respectively annihilation) operators and 1 is the identify. Following the methods of [12] and [6], our phase representation (14)–(16) may thus be easily shown to be related to the phase representation (1). In our representation (14)–(16) any spin Hamiltonian $H(\boldsymbol{S})$ may in fact be transformed into a 'particle Hamiltonian' $\mathcal{H}(\hat{q}, \hat{p})$. In section 4 we have illustrated this by an application to spin-wave theory. In the latter there remains, however, the long-standing open problem of rigorously estimating corrections to the spin-wave picture [13].

Another field of potential application is mesoscopic quantum tunnelling of the magnetization from the point of view of Enz and Schilling [14].

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